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Solving a Class of N-Order Linear Differential Equations by the Recursive Relations and it's Algorithms in MATLAB

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ABSTRACT: In many applied sciences, we find differential equations, which these equations are N-Order Linear differential equations and solutions relatively complex, therefore researchers have forced to use numerical methods, which are contained several errors, i.e. In analysis of computer algorithms, notably in analysis of quicksort and search trees; a number of physics and engineering applications, such as when solving Laplace's equation in polar coordinates; and many other sciences. In this paper, we are using variable change method for the analytic solution a class of N-Order Linear differential equations in problem. Applying a variable change in the equation, then we get conditions where if an equation is also conditions, a simple analytical solution is obtained for it. Because this solution, an exact analytical solution can provide to us, we benefited from the solution of numerical linear differential equations.

Keywords**:** Ordinary Differential Equation, Self-Adjoint Equation, Linear Differential Equations, Wronskian, MATLAB.

INTRODUCTION

In many applied sciences, we find differential equations, which these equations are N-Order Linear differential equations and solutions relatively complex, therefore researchers have forced to use numerical methods, which are contained several errors. (Mohyud-Din, 2009; Allame, 2011; Borhanifar et al., 2011; Sweilam, 2011; Gülsu et al., 2011; Mohyud-Din et al., 2011; Raftari, 2010 and Gondal et al., 2011).

There are several methods for solving equations, there one of which can be seen in the literature (Qiusheng et al., 1996-1994; Demir et al., 2011; Arfken, 1985; Gandarias, 2011; Batiha et al., 2011 and Ugurlu et al., 2011), where the change of variables is very complicated to use.

In this paper, for solving analytical a class of N-Order Linear differential equations, applying a variable change in the equation, then we get conditions where if an equation is also conditions, a simple analytical solution is obtained for it.

Before going to the main point, we start to introduce two following items:

N-Order Liner Ordinary Differential Equation

Homogeneous linear Ordinary differential equation (ODE) of order n is:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0 \tag{1}$$

where $p_{n-1}(x),\ldots, p_1(x), p_0(x)$ continues functions on (a,b) and $p_{n-1}(x)$ is n-1 times differentiable on (a,b).

Equation (1) may be written symbolically as:

$$L(y) = \{D^n + p_{n-1}D^{n-1} + \ldots + p_1 D + p_0\}y = 0 \tag{2}$$

The expression

$$L \equiv D^n + p_{n-1}D^{n-1} + \ldots + p_1 D + p_0$$

is known as a linear differential operator of order n.

Wronskian
The Wronskian of two functions f and g is (Hilderbrand, 1976; Javadpour, 1993 and O'Neil, 1987): $W(x)=W(f,g)=f'g-fg'$.

More generally, for n real- or complex-valued functions $f_1, f_2,..., f_n$, which are n − 1 times differentiable on (a,b), the Wronskian $W(x)=W(f_1,..., f_n)$ as a function on (a,b) is defined by

$$W(x)=\begin{vmatrix} f_1 & \cdots & f_n \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

That is, it is the determinant of the matrix constructed by placing the functions in the first row, the first derivative of each function in the second row, and so on through the (n - 1)st derivative, thus forming a square matrix sometimes called a fundamental matrix.

When the functions f_i are solutions of a linear differential equation, the Wronskian can be found explicitly using Abel's identity, even if the functions f_i are not known explicitly.

Theorem 1*:* if $P(x)y''+Q(x)y'+R(x)y=0$ then $W(x)=e^{-\int\frac{Q}{P}dx}$.

Proof: let two solution of equation by y_1 and y_2, then, since these solutions satisfy the equation, we have

$$Py_1''+Qy_1'+Ry_1=0$$

$$Py_2''+Qy_2'+Ry_2=0$$

Multiplying the first equation by y_2, the second by y_1, and subtracting we find

$$P.(y_1y_2''-y_2y_1'')+Q.(y_1y_2'-y_2y_1')=0$$

Since Wronskian is given by $W=y_1y_2'-y_2y'$ thus $P.\frac{dW}{dx}+Q.W=0$
Solving, we obtain an important relation known as Abel's identity, given by

$$W(x)=e^{-\int\frac{Q}{P}dx}.\blacksquare$$

In general case, for N-Order Liner Ordinary Differential Equation (1), we obtain

$$W(x)=e^{-\int p_{n-1}dx} \qquad (3)$$

SOLVING A CLASS OF N-ORDER LINEAR DIFFERENTIAL EQUATIONS
Theorem2(Main Theorem)*:*If, a homogeneous linear ordinary differential equation of order n :

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+...+p_1(x)y'+p_0(x)y=0$$

verify in the conditions:

$$p_{n-k}=\frac{(n-k+1)}{k.n}\left(np'_{n-k+1}+p_{n-1}p_{n-k+1}\right)$$

where $k=2,3,...,n$.
Then, the answer to equation will be

$$y(x)=e^{-\frac{1}{n}\int p_{n-1}dx}\left(C_{n-1}x^{n-1}+...+C_1x^1+C_0\right).$$

Proof: we show that linear Ordinary differential equation (1) is changeable to two linear differential equations.
By replacing of change variable

$$y=u(x).v(x) \qquad (4)$$

in equation (1),where $u(x)$ and $v(x)$ are continuous and differentiable functions, and

$$y = u.v$$
$$y' = u'.v + u.v'$$
$$y'' = u''.v + 2u'.v' + u.v''$$
$$\vdots$$
$$y^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)}.v^{(k)}$$

After placing in equation (1), we have:

$$\begin{aligned}
&\binom{n}{0} v.u^{(n)} + \binom{n}{1} v'u^{(n-1)} + \binom{n}{2} v''u^{(n-2)} + \binom{n}{3} v'''u^{(n-3)} + \ldots + \binom{n}{n-1} v^{(n-1)}u' + \binom{n}{n} v^{(n)}u \\
&+ p_{n-1}\left(\binom{n-1}{0} vu^{(n-1)} + \binom{n-1}{1} v'u^{(n-2)} + \binom{n-1}{2} v''u^{(n-3)} + \ldots + \binom{n-1}{n-2} v^{(n-2)}u' + \binom{n-1}{n-1} v^{(n-1)}u\right) \\
&+ p_{n-2}\left(\binom{n-2}{0} vu^{(n-2)} + \binom{n-2}{1} v'u^{(n-3)} + \ldots + \binom{n-2}{n-1} v^{(n-1)}u' + \binom{n-2}{n-2} v^{(n-2)}u\right) \\
&\vdots \\
&+ p_2\left(\binom{2}{0} vu'' + \binom{2}{1} v'u' + \binom{2}{2} v''u\right) \\
&+ p_1\left(\binom{1}{0} vu' + \binom{1}{1} v'u\right) \\
&+ p_0 vu \\
&= 0
\end{aligned}$$

After categories, we have:

$$\begin{aligned}
&v.u^{(n)} + \\
&\left[\binom{n}{1} v' + p_{n-1}\binom{n-1}{0} v\right] u^{(n-1)} + \\
&\left[\binom{n}{2} v'' + p_{n-1}\binom{n-1}{1} v' + p_{n-2}\binom{n-2}{0} v\right] u^{(n-2)} + \\
&\left[\binom{n}{3} v''' + p_{n-1}\binom{n-1}{2} v'' + p_{n-2}\binom{n-2}{1} v' + p_{n-3}\binom{n-3}{0} v\right] u^{(n-3)} + \\
&\vdots \\
&\left[\binom{n}{n-1} v^{(n-1)} + p_{n-1}\binom{n-1}{n-2} v^{(n-2)} + \cdots + p_2\binom{2}{1} v' + p_1\binom{1}{0} v\right] u' + \\
&\left[\binom{n}{n} v^{(n)} + p_{n-1}\binom{n-1}{n-1} v^{(n-1)} + p_{n-2}\binom{n-2}{n-2} v^{(n-2)} + \cdots + p_1\binom{1}{1} v' + p_0 v\right] u = 0
\end{aligned}$$

Or

$$v.u^{(n)} + \sum_{k=1}^{n}\left[\sum_{i=0}^{k} \binom{n-i}{k-i} p_{n-i} v^{(k-i)}\right] u^{(n-k)} = 0 \tag{5}$$

Where $p_n(x) = 1$.

Now, we assume that the coefficient $u^{(n-1)}$ is zero and we can calculate the value $v(x)$, i.e.

$$\binom{n}{1} v' + p_{n-1}\binom{n-1}{0} v = 0$$
$$\Rightarrow v' = -\frac{1}{n} p_{n-1} v$$
$$\Rightarrow v = e^{-\frac{1}{n}\int p_{n-1}dx} \tag{6}$$

Now, corresponding to equation (3) we have

$$v(x) = e^{-\frac{1}{n}\int p_{n-1}dx} = \sqrt[n]{W(x)} \tag{7}$$

Now, using Equation (5), (7) and assuming that the coefficients $u^{(n-2)}, u^{(n-3)}, ..., u', u$ are zero, we get the condition that if an equation is also conditions, can be solved with this method:

$$\binom{n}{2} v'' + p_{n-1}\binom{n-1}{1} v' + p_{n-2}\binom{n-2}{0} v = 0$$

$$\Rightarrow \left(-\frac{n-1}{2n} p_{n-1}^2 - \frac{n-1}{2} p'_{n-1} + p_{n-2} \right) v = 0$$

$$\boxed{p_{n-2} = \frac{(n-1)}{2!.n}\left(np'_{n-1} + p_{n-1}^2 \right)} \quad (8)$$

And

$$\binom{n}{3} v''' + p_{n-1}\binom{n-1}{2} v'' + p_{n-2}\binom{n-2}{1} v' + p_{n-3}\binom{n-3}{0} v = 0$$

$$\boxed{p_{n-3} = \frac{(n-1)(n-2)}{3!.n^2}\left(n^2 p''_{n-1} + 3np_{n-1}p'_{n-1} + p_{n-1}^3 \right)} \quad (9)$$

And

$$\boxed{p_{n-4} = \frac{(n-1)(n-2)(n-3)}{4!.n^3}\left(n^3 p'''_{n-1} + 3n^2 p'^2_{n-1} + 4n^2 p_{n-1}p''_{n-1} + 6np_{n-1}^2 p'_{n-1} + p_{n-1}^4 \right)}$$

In general, we can obtain the following recursive relation:

$$\boxed{p_{n-k} = \frac{(n-k+1)}{k.n}\left(np'_{n-k+1} + p_{n-1}p_{n-k+1} \right)} \quad (10)$$

Where $k = 2,3,...,n$.

- ✓ So if a linear differential equation in Equation (10) and the above assumptions is true, according to equation (5) we have:

$$v.u^{(n)} = 0 \qquad \Rightarrow \; u^{(n)} = 0$$

Or

$$u(x) = C_{n-1}x^{n-1} + ... + C_1x^1 + C_0 \quad (11)$$

Where $C_{n-1}, ..., C_1, C_0$ are arbitrary real numbers.

So, we have from Equations (4), (7) and (11):

$$v(x) = e^{-\frac{1}{n}\int p_{n-1}dx} = \sqrt[n]{W(x)}$$

$$u(x) = C_{n-1}x^{n-1} + ... + C_1x^1 + C_0 \quad (12)$$

And, the answer to linear differential equation (1) will be

$$\boxed{y(x) = v(x).u(x) = e^{-\frac{1}{n}\int p_{n-1}dx}\left(C_{n-1}x^{n-1} + ... + C_1x^1 + C_0 \right)} \;\blacksquare \quad (13)$$

EXAMPLES and APPLICATIONS

EX.1) Solve the equation

$$(1-x^2)y'' - 2xy' - \frac{1}{1-x^2}y = 0 \quad (14)$$

***Solution*:** By virtue of equation (1) we have

$$p_1(x) = \frac{-2x}{1-x^2} \quad , \quad p_0(x) = \frac{-1}{(1-x^2)^2}$$

Obviously, that equation (8) is established, i.e.

$$p_0 = \frac{(2-1)}{2!.2}\left(2p'_1 + p_1^2 \right)$$

So, we have:

$$v(x)=e^{-\frac{1}{2}\int p_1 dx}=e^{-\frac{1}{2}\int\frac{-2x}{1-x^2}dx}=e^{-\frac{1}{2}Ln(1-x^2)}=\frac{1}{\sqrt{1-x^2}}$$

$$u(x)=C_1x+C_0$$

And, the answer to linear differential equation (14) will be

$$y(x)=v(x).u(x)=\frac{1}{\sqrt{1-x^2}}\left(C_1x+C_0\right)$$

Analytical and very simple answer is obtained, while using the series, is a long way.

EX.2) Solve the equation

$$y'''+xy''+\frac{3+x^2}{3}y'+\frac{9x+x^3}{27}y=0 \qquad (15)$$

Solution: By virtue of equation (1) we have

$$p_2(x)=x\ ,\ p_1(x)=\frac{3+x^2}{3}\ \ ,\ \ p_0(x)=\frac{9x+x^3}{27}$$

Obviously, that equations (8) and (9) are established, i.e.

$$p_1=\frac{(3-1)}{2!.3}\left(3p_1'+p_1^2\right)$$

$$p_0=\frac{(3-1)(3-2)}{3!.3^2}\left(3^2p_2''+3.3.p_2p_{n-1}'+p_2^3\right)$$

So, we have:

$$v(x)=e^{-\frac{1}{3}\int p_2 dx}=e^{-\frac{1}{3}\int x dx}=e^{-\frac{1}{6}x^2}$$

$$u(x)=C_2x^2+C_1x^1+C_0$$

And, the answer to linear differential equation (15) will be

$$y(x)=v(x).u(x)=e^{-\frac{1}{6}x^2}\left(C_2x^2+C_1x+C_0\right)$$

Analytical and very simple answer is obtained, while using the series, is a long way.

EX.3) Solve the equation

$$e^{-x}y''+y'+\frac{2+e^x}{4}y=0 \qquad (16)$$

Solution: By virtue of equation (1) we have

$$p_1(x)=e^x\ \ ,\ \ p_0(x)=\frac{e^x(2+e^x)}{4}$$

Obviously, that equation (8) is established, i.e.

$$p_0=\frac{(2-1)}{2!.2}\left(2p_1'+p_1^2\right)$$

So, we have:

$$v(x)=e^{-\frac{1}{2}\int p_1 dx}=e^{-\frac{1}{2}\int e^x dx}=e^{-\frac{1}{2}e^x}$$

$$u(x)=C_1x^1+C_0$$

And, the answer to linear differential equation (16) will be

$$y(x)=v(x).u(x)=e^{-\frac{1}{2}e^x}\left(C_1x+C_0\right)$$

Analytical and very simple answer is obtained, while using the series, is a long way.

EX.4) Solve the equation

$$\frac{d}{dx}\left(\alpha.x^ny'\right)+\frac{\alpha.n.(n-2)}{4}x^{n-2}y=0 \qquad (17)$$

Where α, n are constants and $\alpha \neq 0$ [4, 5 and 6].
Solution: By virtue of equation (1) we have

$$p_1(x) = \frac{n}{x} \quad , \quad p_0(x) = \frac{n(n-2)}{4x^2}$$

Obviously, that equation (8) is established, i.e.

$$p_0 = \frac{(2-1)}{2!.2}\left(2p_1' + p_1^2\right)$$

So, we have:

$$v(x) = e^{-\frac{1}{2}\int p_1 dx} = e^{-\frac{1}{2}\int \frac{n}{x} dx} = e^{-\frac{n}{2} Ln(x)} = x^{-\frac{n}{2}}$$

$$u(x) = C_1 x^1 + C_0$$

And, the answer to linear differential equation (17) will be

$$y(x) = v(x).u(x) = x^{-\frac{n}{2}}\left(C_1 x + C_2\right)$$

Analytical and very simple answer is obtained, while using the series, is a long way.
EX.5) Solve the equation

$$y'' + 2y' + y = 0 \tag{18}$$

Solution: By virtue of equation (1) we have

$$p_1(x) = 2 \quad , \quad p_0(x) = 1$$

Obviously, that equation (8) is established, and we have:

$$v(x) = e^{-\frac{1}{2}\int p_1 dx} = e^{-\frac{1}{2}\int 2 dx} = e^{-x} \quad , \quad u(x) = C_1 x + C_0$$

And, the answer to linear differential equation (18) will be

$$y(x) = v(x).u(x) = e^{-x}\left(C_1 x + C_0\right)$$

Analytical and very simple answer is obtained.

EX.6) In the special case, if $p_{n-1}(x) = 1$, then we have

$$p_{n-k} = \frac{(n-k+1)}{k.n}\left(p_{n-k+1}\right) = \binom{n}{k}\frac{1}{n^k}$$

where $k = 2,3,...,n$.
And, the answer to linear differential equation (1) will be

$$y(x) = e^{-\frac{1}{n}x}\left(C_{n-1}x^{n-1} + ... + C_1 x^1 + C_0\right).$$

ALGORITHMS IN MATLAB

Algorithm for Finding the Functions

Algorithm for finding the functions in equation (10) are true, is as follows:
First, in MATLAB $\Rightarrow$ Menu "File" $\Rightarrow$ Select "NEW" $\Rightarrow$ Select "M-File" $\Rightarrow$ The following function to copy in file, and then save the file to "norder_find.m" name:

```
function norder_find(f)
syms n k
n=f{2};

for k=2:n
  f{k}=(((n-k+1)/(n*k))*((n*diff(f{k-1}))+(f{1}*f{k-1})));
  disp(sprintf('p(%d)', n-k));
  display(simplify(f{k}));
end
end
```

Secondly, in the window "Command Windows" of MATLAB, type the following commands:

```
>> syms x
>> f{1} = "function: p_{n-1}(x)";
>> f{2} = "order differential equation: n";
>> norder_find(f)
```

EX.7) in the window "Command Windows" of MATLAB, type the following commands:

```
>> syms x
>> f{1} = x^2;
>> f{2} =2;
>> norder_find(f)
```

Then, key "ENTER"

```
p(0)
=
x+1/4*x^4
```

Thus, functions that are true in the equation (10), are calculated, the desired equation is as follows:

$$y'' + x^2 y' + \frac{4x + x^4}{4} y = 0$$

i.e.

$$y(x) = e^{-\frac{1}{6}x^3}\left(C_1 x + C_0\right).$$

Algorithm to check the true functions

Algorithm to check the true functions in equation (10), is as follows:

First, in MATLAB $\Rightarrow$ Menu "File" $\Rightarrow$ Select "NEW" $\Rightarrow$ Select "M-File" $\Rightarrow$ The following function to copy in file, and then save the file to "norder_test.m" name:

```
function norder_test(f)
syms n k
n=size(f,2);

k=2;
flag=1;
while(flag==1 && k<=n)
      if (simplify(f{n-k+1}-(((n-k+1)/(n*k))*((n*diff(f{n-k+2}))
                          +(f{n}*f{n-k+2}))))==0)
          flag=1;
          k=k+1;
      else
          flag=0;
      end
end
if (flag==1)
   display('Yes');
else
   display('No');
end
end
```

Secondly, in the window "Command Windows" of MATLAB, type the following commands:

```
>> syms x
>> f{1}="the intended function : p_0(x)";
>> f{2}="the intended function : p_1(x)";
>>…
```

>> *f{n}="the intended function : $p_{n-1}(x)$";*
>> *norder_test(f)*

EX.8) in the window "Command Windows" of MATLAB, type the following commands:
>> *syms x*
>> *f{1}=1/2*exp(x)+1/4*exp(2*x);*
>> *f{2}=exp(x);*
>> *norder_test(f)*

Then, key "ENTER"
>> *yes*

Thus, if $p_1(x)=e^x$, $p_0(x)=\frac{1}{2}e^x+\frac{1}{4}e^{2x}$, equation (10) are established, i.e. equation

$$y''+e^x y'+(\frac{1}{2}e^x+\frac{1}{4}e^{2x})y=0$$

is the answer to:

$$y(x)=e^{-\frac{1}{2}e^x}\left(C_1x+C_0\right).$$

CONCULSION

The governing equation for stability analysis of a variable cross-section bar subject to variably distributed axial loads, dynamic analysis of multi-storey building, tall building and other systems are written in the form of a unified linear differential equation of the second order.
The key step in transforming the unified equation is the equation (10).
Many difficult problems in the field of static and dynamic mechanics are solved by the unified equation proposed in this paper.

REFERENCES

Allame M, Azad N. 2011. Solution of Third Order Nonlinear Equation by Taylor Series Expansion, World Applied Sciences Journal 14 (1): 59-62, ISSN 1818-4952.
Arfken G. 1985. "Self-Adjoint Differential Equations." §9.1 in Mathematical Methods for Physicists, 3rd ed. Orlando, FL: Academic Press, pp. 497-509.
Batiha K, Batiha B. 2011. A New Algorithm for Solving Linear Ordinary Differential Equations, World Applied Sciences Journal 15 (12): 1774-1779, ISSN 1818-4952.
Borhanifar A, Kabir M M, HosseinPour A. 2011. A Numerical Method for Solution of the Heat Equation with Nonlocal Nonlinear Condition, World Applied Sciences Journal 13 (11): 2405-2409, ISSN 1818-4952.
Delkhosh M. 2011. The Conversion a Bessel's Equation to a Self-Adjoint Equation and Applications, World Applied Sciences Journal 15 (12): 1687-1691, ISSN 1818-4952.
Demir D, Bildik N, Konuralp A, Demir A. 2011. The Numerical Solutions for the Damped Generalized Regularized Long-wave Equation by Variational Method, World Applied Sciences Journal 13 (Special Issue of Applied Math): 08-17, ISSN 1818-4952.
Gandarias M. 2011. Weak self-adjoint differential equations, J. Phys. A: Math. Theor. 44. 262001 (6pp).
Gondal M A, Arife A S, Majid Khan and Iqtadar Hussain. 2011. An Efficient Numerical Method for Solving Linear and Nonlinear Partial Differential Equations by Combining Homotopy Analysis and Transform Method, World Applied Sciences Journal 14 (12): 1786-1791, ISSN 1818-4952.
Gülsu M, Öztürk Y, Sezer M. 2011. Numerical Solution of Singular Integra-differential Equations with Cauchy Kernel, World Applied Sciences Journal 13 (12): 2420-2427, ISSN 1818-4952.
Hilderbrand F B. 1976. Advanced Calculus for Applications (2-ed), New Jersey.
Javadpour S H, Alavi. 1993. an Introduction to Ordinary and Partial Differential Equations, Iran.
Mohyud-Din S T. 2009. Solutions of Nonlinear Differential Equations by Exp-function Method, World Applied Sciences Journal 7 (Special Issue for Applied Math): 116-147, ISSN 1818-4952.
Mohyud-Din T, Yildirim A, Berberler M, Hosseini M. 2010. Numerical Solution of Modified Equal Width Wave Equation, World Applied Sciences Journal 8 (7): 792-798, ISSN 1818-4952.
Olver F, Maximon L, Bessel function. 2010. NIST Handbook of Mathematical Functions, Cambri- dge University Press, ISBN 978-0521192255, MR2723248.
O'Neil P V. 1987. Advanced Engineering Math-ematics (2-ed), California, Wadsworth.
Qiusheng L, Hong C, Guiqing L. 1996. Static and Dynamic analysis of straight bars with variable cross-section, Computer & Structures Vol. 59, No. 6, p.1185-1191.

Qiusheng L, Hong C, Guiqing L. 1994. Analysis of Free Vibrations of Tall Buildings, Journal of Engineering Mechanics, Vol. 120, No. 9, pp. 1861-1876.

Raftari B. 2010. Numerical Solutions of the Linear Volterra Integro-differential Equations: Homotopy Perturbation Method and Finite Difference Method, World Applied Sciences Journal 9 (Special Issue of Applied Math): 07-12, ISSN 1818-4952.

Sweilam N H, Nagy A M. 2011. Numerical Solution of Fractional Wave Equation using Crank-Nicholson Method, World Applied Sciences Journal 13 (Special Issue of Applied Math): 71-75, ISSN 1818-4952.

Teukolsky S, Vetterling WT. 2007. Section 6.5. Bessel Functions of Integer Order, Numerical Recipes: The Art of Scientific Computing (3rd ed.), New York: Cambridge University Press, ISBN 978-0-521-88068-8.

Ugurlu Y, Inan I E, Kilic B. 2011. Analytic solutions of some partial differential equations by using homotopy perturbation method, World Applied Sciences Journal 12 (11): 2135-2139, ISSN 1818-4952.